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Hypercubic lattice SAW exponents ν and γ: 3.99 dimensions revisited

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Abstract. The self-avoiding walk (SAW) exponents ν and γ are computed over a range of dimensions ($1 \leqslant d < \infty$) from *exact* expressions for the mean-square end-to-end distance $\langle R_n^2 \rangle$ and the partition function Q_n of SAWs having a limited number of steps, $n \leqslant 11$. SAW exponents (ν, γ) for *arbitrary* dimension d are estimated by applying standard extrapolation techniques to our direct enumeration data which has been analytically continued to variable dimension. Exponent estimates obtained from continuum theories of self-avoiding paths are compared with the SAW calculations.

1. Introduction

It is well known that the geometrical properties of self-avoiding and random walks exhibit a strong dependence on spatial dimensionality d and that the 'universal' critical behaviour in many systems undergoing phase transitions is intimately related to the geometrical properties of these walks. Specifically, we mention Symanzik's formulation [1] of ϕ^4 field theory in terms of a 'gas' of interacting Brownian paths, Domb's calculation of $O(m)$ lattice spin model properties in terms of interacting SAWs [2] and De Gennes' discovery of an exact relation between the $m \to 0$ limit of the $O(m)$ model and SAWs [3]. The accurate characterization of the geometrical properties of SAWs is consequently a problem with many practical physical applications, besides the rather obvious applications to the solution properties of polymers [4, 5].

Many analytical and numerical studies of self-avoiding walks with nearest-neighbour interactions on a variety of lattices have appeared since the pioneering studies by Orr [6] and Fisher *et al* [7]. Recent works [8] often emphasize the calculation of the SAW 'critical indices' ν and γ, and the 'connectivity constant' μ. Rigorous results include a proof of the existence [9, 10] of μ and the relation [11], $\gamma = 2\nu = 1$, $d \geqslant 5$. Conformal invariance calculations in $d = 2$ suggest the exact results [12] $2\nu = \frac{3}{2}$ and $\gamma = \frac{43}{32}$.

The study of lattice SAW models has developed in parallel to analytic theories of self-avoiding paths based on $O(m \to 0)$ field theoretic methods [3] or direct formulations in terms of Wiener path-integration [4, 5, 13]. Application of the perturbative Wilson–Fisher ε-expansion method has provided SAW information that is complementary to lattice model studies [5, 13]. Simple dimensional analysis in the continuum theory of self-avoiding paths indicates the singular role of $d=4$ dimensions and the $\varepsilon=4-d$ perturbation theory leads to apparently accurate and unique estimates of the exponents (γ, ν) and dimensionless amplitude ratios [5, 13, 14]. These formal calculations provide support for the hypothesis of 'universality' [existence and uniqueness of exponents (γ, ν)] which is conventionally *assumed* in the numerical interpretation of lattice SAW data.

Given the success of continuum model calculations for the geometrical properties of self-avoiding paths as a function of d, there have been remarkably few lattice studies which directly treat the spatial dimension as variable in the investigation of large scale geometrical properties of SAWs. However, there are formal $1/d$ expansion calculations of the 'non-universal' constant μ and some other geometrical properties of SAWs [15, 16]. There have also been some interesting studies of SAWs on 'fractal' lattices [17, 18], which are thought to roughly interpolate between regular lattices of integer dimension. SAWs on percolation clusters have also been considered [19, 20] as idealized models of polymers in a disordered environment. Our present analytical continuation of lattice enumeration data to continuous d indicates that the idea of relating fractal lattices to regular lattices in non-integer dimensions has some heuristic value, even if a quantitative relation is difficult to establish. Le Guillou and Zinn-Justin similarly examine the relation between Ising model critical exponents for fractal lattices and $O(m=1)$ Borel resummation predictions in variable dimension [21].

Two previous papers [22, 23] employ direct enumeration data in variable dimension $(d=1, 2, \ldots)$ to obtain *exact* expressions for the partition function C_n and the mean-square end-to-end distance $\langle \boldsymbol{R}_n^2 \rangle$ for nearest-neighbour interacting SAWs in continuously variable dimension. These initial calculations are limited to rather modest walk lengths $(n \leqslant 11)$, but improved computer resources and more efficient coding should allow a systematic extension to longer 'chains'.

Abe [24] and Baker and Benofy [25] discuss a similar extension of the Ising model to continuous spatial dimension d. Baker and Benofy show that this analytic continuation is 'equivalent' to the Wilson–Fisher ε-analytic continuation [3, 25]. However, they do not investigate the variation of the Ising exponents (γ, ν) with dimension. Our previous calculations [22, 23] of C_n and $\langle \boldsymbol{R}_n^2 \rangle$ for SAWs in variable dimension are used here in conjunction with standard ratio and approximant methods to evaluate the exponents γ and ν. These estimates are then compared with predictions of RG ε-expansion, Borel resummation and Flory (self-consistent field) calculations in non-integer dimensions [3]. Thus, self-avoiding lattice walks in $d=3.99$ dimensions [3] *can be investigated by our methods as a model without mathematical ambiguity.* The advantage of the present non-perturbative calculations is that the results are not restricted to dimensions 'near' four.

2. Exponents ν and γ as a function of dimension

Orr [6] initiated exact SAW *enumeration for walk lengths n long enough to yield* non-trivial estimates of their statistical properties, and Fisher *et al* [7] significantly extended this direct approach. In these pioneering studies Orr [6] introduced the basic notion of a 'connectivity constant' μ (free energy per step) and Fisher *et al* [7] defined 'critical exponents' for SAWs. Rigorous calculations later established these ideas [9–12]. Our recent work provides a further extension of these classical enumeration studies to include variable spatial dimensions [22, 23].

We begin with the direct enumeration expressions for C_n and $\langle \boldsymbol{R}_n^2 \rangle$ of nearest-neighbour interacting SAWs with $n \leqslant 11$, as developed in previous papers [22, 23]. C_n is defined as a weighted sum,

$$C_n(d, \eta) = \sum_{m=0} C_{n,m}\eta^m \qquad \eta = e^{\omega} \qquad \omega = -\varepsilon/k_B T \tag{1}$$

where $C_{n,m}$ is the number of SAW configurations which have m-nearest-neighbour contacts, η is a Boltzmann factor for a nearest-neighbour interaction with energy ε, and $k_B T$ defines the energy units. The zero energy state corresponds to an equal weighting of all SAWs ($\eta = 1$). More compact SAW configurations are weighted for walks with attractive interactions, $\eta > 1$. The sum in (1) extends to the maximum number of nearest-neighbour contacts [6, 7] for a SAW on a hypercubic lattice. $\langle \boldsymbol{R}_n^2 \rangle$ is likewise represented as a weighted sum,

$$\langle \boldsymbol{R}_n^2 \rangle = \left(\sum_{m=0} r_{n,m}\eta^m \right) \Big/ C_n \tag{2}$$

where $r_{n,m}$ and $C_{n,m}$ for $n \leqslant 11$ are tabulated in [22] and [23] respectively, for integer dimensions in the range $2 \leqslant d \leqslant 6$. The coefficients $r_{n,m}$ and $C_{n,m}$ are expressed as polynomials in the spatial dimension [22, 23], so that the lattice enumeration data can be extended formally to *continuously variable* dimension. This analytic continuation is equivalent to that employed in continuum ε-expansion calculations [25].

The estimation of the exponents ν and γ requires some knowledge of the limiting variation of $\langle \boldsymbol{R}_n^2 \rangle$ and C_n. Formal RG calculations and numerical evidence suggest the asymptotic behaviour [26–28],

$$\langle \boldsymbol{R}_n^2 \rangle = A_R n^{2\nu}[1 + B_R/n + C_R n^{-\Delta} + \ldots] \tag{3}$$

$$C_n = A_c n^{\gamma-1}\mu^n[1 + B_c/n + C_c n^{-\Delta} + \ldots]. \tag{4}$$

Our lattice model estimates of ν and γ *presume* the validity of (3) and (4). Formal calculations in $d = 2$ indicate [12],

$$2\nu(d = 2) = 1.5 \qquad \gamma(d = 2) = \tfrac{43}{32} \tag{5}$$

in consistency with the scaling assumed in (3) and (4). There is no rigorous proof of the existence and uniqueness of the exponents (γ, ν) below $d = 5$ except for $d = 1$.

Numerical estimates of SAW exponents ν and γ are obtained by applying some standard variations on the ratio method [7, 8, 30]. Ishinabe discusses these methods in a detailed assessment of exact enumeration estimates of SAW exponents for $d = 2$ and $d = 3$ [31]. Ratio extrapolant methods are directly applied to our analytic expressions for variable dimension as in former analyses restricted to integer dimensions. We first estimate ν and γ following a conventional ratio method with associated Neville tables. C_R and Δ in (3) are determined for this trial ν as discussed in [31]. (This procedure was ineffective for $\langle \boldsymbol{R}_n^2 \rangle$ in $d = 2.75$ and 4, and for C_n in all d.) We then obtain an improved estimate of ν by repeating the first process using the transformed series [31] $R_n^* = \langle \boldsymbol{R}_n^2 \rangle / \{1 + C_R n^{-\Delta}\}$. The analytic corrections to scaling B_R and B_c in (3) and (4) can be *calculated* [22, 23] by exploiting our previous $1/d$ expansions of $\langle \boldsymbol{R}_n^2 \rangle$ and C_n. Inclusion of these terms is restricted to the range $4 < d \leqslant 6$ since this is the range where such corrections are expected from dimensional analysis to be comparable to the non-analytic corrections and $1/d$ expansions should be applicable in this range. We also estimate ν and γ using the transformed series including only analytic corrections $C_n^* = C_n/[1 + B_c/n]$. Estimates of ν and γ with or without such corrections are also presented in table 1. The dimensional dependence of the exponent Δ in equations (3) and (4) will be discussed elsewhere along with the dimension and interaction dependence of μ.

(A) The exponent ν

Figure 1 exhibits a regular decrease of our lattice model estimates of ν in the range $2 < d < 4$, while ν for $d > 4$ approaches the constant value $\frac{1}{2}$. This figure suggests that ν is *piecewise analytic* in the intervals $2 < d < 4$ and $4 < d < \infty$. The spherical model correlation length exponent ν_s and the susceptibility exponent γ_s are known to be

Table 1

	Ratio Method Estimates of ν		
d	no correction	Δ correction	$(\Delta + 1/d)$ correction
2†	0.7490 ± 0.0014	0.7505 ± 0.0010 (○)	
2.75	0.63 ± 0.003 (○)		
3‡	0.5920 ± 0.0004	0.5898 ± 0.0002 (○)	
3.25	0.565 + 0.008	0.555 ± 0.005 (○)	
3.5	0.545 ± 0.005	0.535 ± 0.005 (○)	
4§	0.500 ± 0.004 (○)		
4.5	0.514 ± 0.002	0.510 ± 0.0015 (○)	0.505 ± 0.0015 (□)
5	0.507 ± 0.001	0.5040 ± 0.0006 (○)	0.5006 ± 0.0006 (□)
5.5	0.5040 ± 0.0008	0.5001 ± 0.0005 (○)	0.4997 ± 0.0003 (□)
			$1/d$ correction
6	0.5006 ± 0.0006 (○)		0.5002 ± 0.0005 (□)
6.5	0.4996 ± 0.0005 (○)		0.4995 ± 0.0004 (□)
7	0.4993 ± 0.0003 (○)		0.4993 ± 0.0005 (□)

	Ratio Method Estimates of γ	
d	no correction	$1/d$ correction
2†	1.342 ± 0.002 (○)	
2.75	1.23 ± 0.07 (○)	
3‡	1.162 ± 0.006 (○)	
3.25	1.11 ± 0.015 (○)	
3.50	1.07 ± 0.015 (○)	
3.75	1.05 ± 0.015 (○)	
4‖	1.00 ± 0.015 (○)	1.055 ± 0.008 (□)
4.25	1.040 ± 0.008 (○)	1.0328 ± 0.005 (□)
4.5	1.035 ± 0.005 (○)	1.0104 ± 0.005 (□)
5	1.025 ± 0.005 (○)	1.0008 ± 0.005 (□)
5.5	1.016 ± 0.005 (○)	
6	1.007 ± 0.005 (○)	
6.5	1.003 ± 0.004 (○)	
7	1.002 ± 0.004 (○)	

† $n \leq 22$ [50].
‡ $n \leq 14$, unpublished direct enumeration data $C_{n,m}$.
§ Includes log correction $\delta = \frac{1}{4}$ as indicated in [43, 44].
‖ $n \leq 13$, $\delta = \frac{1}{4}$ [44].

piecewise analytic [32],

$$\gamma_s = 2\nu_s = 2/(2-\varepsilon) \qquad \varepsilon = 4-d \qquad 2<d<4 \tag{6a}$$

$$\gamma_s = 2\nu_s = 1 \qquad d>4 \tag{6b}$$

and our suggestion of piecewise analytic SAW exponents is further supported by Hara and Slade's proof [11] that the SAW exponents take their mean-field values for $d \geqslant 5$,

$$\gamma = 2\nu = 1 \qquad d \geqslant 5. \tag{7}$$

Flory self-consistent field (SCF) calculations [33-36] also yield a piecewise analytic variation of ν with dimension,

$$\nu_F(1 \leqslant d < 4) = 3/(d+2) \equiv 1/(2-\varepsilon/3) \tag{8a}$$

$$\nu_F = \tfrac{1}{2} \qquad d \geqslant 4. \tag{8b}$$

The formal RG ε-expansion [3, 5] about $d=4$ (which is probably a point of non-analyticity) indicates that 2ν has the asymptotic expansion,

$$2\nu_{RG}(d \leqslant 4) \sim 1 + (\varepsilon/8) + \tfrac{15}{4}(\varepsilon/8)^2 + O(\varepsilon^3). \tag{9}$$

Second order in ε has been suggested as the optimal order of truncation for polymer excluded volume [37], so that the estimate of 2ν should become *worse* upon inclusion of higher order terms in ε in (9). Proof that the series (9) is of the Stietjes-type would rigorously justify the application of the optimal order truncation criterion [37, 38]. Borel resummation and other resummation methods [29, 39-41] provide more precise estimates of ν for the physically interesting case $d=3$ and these estimates are mentioned below in our discussion of improved estimates of ν as a function of dimension.

Comparison of the Flory SCF calculation (8) and lattice model estimates of ν in figure 1 yield good qualitative agreement over the entire range of d. The Flory SCF calculation is exact for $d=1$ and $d>5$ [11] and equation (8) is thought to be exact in $d=2$ [12]. However, equation (8) appears to deviate systematically from the lattice model estimates for $d \approx 4^-$. Such a deviation is expected from RG ε-expansion theory which should have its greatest accuracy for $\varepsilon \simeq 0^+$. Indeed, the RG ε-expansion for ν agrees rather well near $d=4$ with the lattice model calculations in figure 1. Equations (8) and (9) imply the RG theory yields a slower variation of ν with dimension than the Flory SCF theory. This is manifest from the limits,

$$\lim_{\phi \to 0^+} (2\nu_{RG} - 1)/\phi = \tfrac{1}{4} \qquad \phi \equiv \varepsilon/2 \tag{10a}$$

$$\lim_{\phi \to 0^+} (2\nu_F - 1)/\phi = \tfrac{1}{3}. \tag{10b}$$

The limit (10a) determines the exponent δ governing the logarithmic corrections at the critical dimension [43, 44], $\varepsilon \to 0^+$,

$$\langle R_n^2 \rangle = A_R n [\ln n]^\delta \qquad \delta = \tfrac{1}{4} \qquad n \to \infty. \tag{10c}$$

The log correction in (10c) is included in the computation of ν for $d=4$ in table 1. In conclusion, the Flory SCF calculation agrees better with our lattice SAW calculations in low dimensions $1 \leqslant d \leqslant 2$ while the RG ε-expansion calculations agree better for dimensions near $d=4$. Evidence supporting a critical dimension $d_c = 4$ is clearly exhibited by figure 1, which accords with previous findings by Abe [24] for the Ising model on a hypercubic lattice.

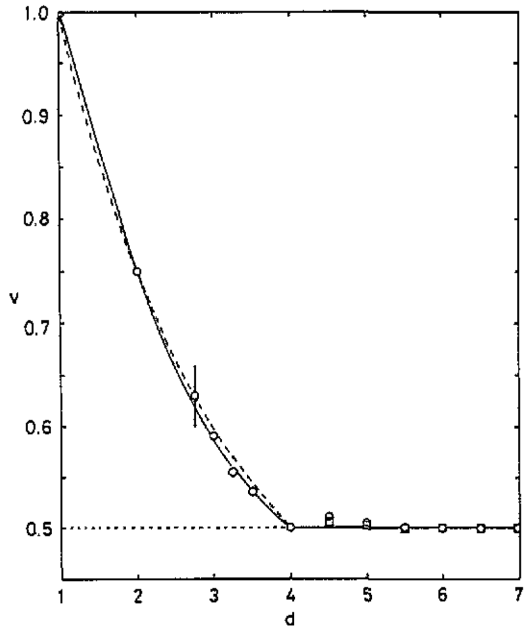

Figure 1. Exponent ν as a function of dimension. Solid curve indicates equation (11) and dashed curve the Flory estimate equation (8). □ data includes analytic correction to scaling while ○ data does not include such corrections. See table 1 and text.

An improved estimate of ν over the range $1 \leqslant d < 4$ can be obtained by requiring the approximant for ν to be consistent with established values in $d = 1$ and 2 and with RG results for dimensions near $d = 4$. A simple Lagrange interpolation polynomial combining this information is given by

$$\phi/(2\nu - 1) = 4[1 - 2\phi/3 + \phi^2/6] \qquad 1 \leqslant d < 4 \tag{11a}$$

$$\nu = \tfrac{1}{2} \qquad d \geqslant 4. \tag{11b}$$

Interestingly, equation (11) produces $\nu(d = 3) = \frac{10}{17} = 0.5882$—in excellent agreement with Borel resummation estimates,

$$\nu(d = 3) = 0.5880 \pm 0.001 \text{ [21, 39]}$$

$$\nu(d = 3) = 0.5885 \pm 0.0025 \text{ [40]}$$

$$\nu(d = 3) = 0.5886 \text{ [41]}.$$

Direct enumeration [34], Monte Carlo lattice SAW data [42] and experimental data for real polymer chains in good solvents [5, 13, 39] are consistent with ν close to 0.59 in $d = 3$. Such a value of ν also agrees with the second order ε-expansion calculation of ν (see equation (9)) $\nu_{RG}(d = 3) = 0.592$. The ε-expansion estimate of ν, however, degrades at higher order [37].

Equation (11) agrees very well with the lattice computations presented in figure 1 for $1 \leqslant d \leqslant 4$. However, our calculations for $4 \leqslant d \leqslant 6$ tend to be slightly higher than $\nu = \frac{1}{2}$. Inclusion of the analytic correction to scaling B_R (see equation 3) produces exponent estimates closer to $\nu = \frac{1}{2}$. Further study of corrections to scaling in this range of dimensions is needed for a more accurate estimation of exponents. The situation of d slightly above $d=4$ seems to be very delicate.

(B) The exponent γ

The SAW exponent γ displays a more complicated dependence on dimension than the exponent ν. Figure 2 shows that $\gamma \approx 2\nu$ in the range $3 \leqslant d < 4$, but γ passes through a *maximum* near $d=2$ and *decreases* to $\gamma = 1$ in $d=1$. We are unaware of a previous discussion of this maximum. However, in retrospect, the non-monotone variation of γ is obvious. When $d=1$, it trivially follows that $\gamma=1$ since a fully extended chain has a number of possible configurations which is independent of chain length. Hara and Slade [11] prove that $\gamma=1$ in high dimensions $d \geqslant 5$ and RG theory indicates that $\gamma > 1$ for $d<4$ and $\varepsilon \approx 0^+$ [3, 5]. Thus, if γ is assumed to vary continuously with dimension then γ must possess at least one finite maximum in the range $1 \leqslant d < 4$.

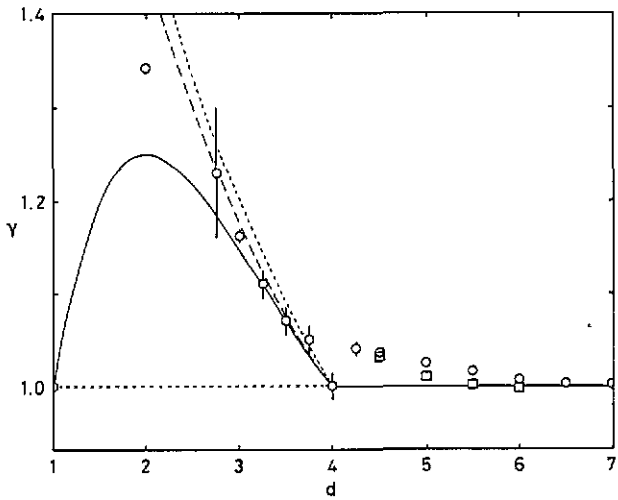

Figure 2. Exponent γ as function of dimension. Solid curve from equation (14) and the long and short dashed curves indicate the RG and Flory-type predictions, respectively. □ and ○ have same meaning as in figure 1.

Unfortunately, it is rather difficult to locate precisely the position of the maximum in γ from our calculations. The maximum occurs somewhere in the vicinity of $d=2$. Ratio method extrapolations of the SAW exponents for our rather short chains tend to become rather erratic in the interval $(1.5 < d < 2.5)$, except for $d=2$. We avoid including non-integer dimension exponent estimates in this range until a more appropriate extrapolation method is developed. Baker and Benofy [25] suggest that the analytical continuation of lattice model computations to non-integer dimensions inherently involves a kind of 'frustration' in the interactions, and they even suggest that the radii of convergence of the high temperature (specifically Ising) series could shrink to zero for non-integer dimensions! Thus, the erratic extrapolations of γ and ν in the range $1.5 < d < 2.5$ deserve careful examination. No difficulties are encountered in our extrapolations for the range $3 \leqslant d$ and for $d=2$.

RG ε-expansion calculations of the SAW exponent γ give no indication of a maximum,

$$\gamma_{\mathrm{RG}} \sim 1+\varepsilon/8+\tfrac{13}{4}(\varepsilon/8)^2+O(\varepsilon^3) \qquad \varepsilon \approx 0^+. \tag{12a}$$

Such a maximum is also absent in the recent Flory–Pietronero [42] calculation of γ,

$$\gamma_{\mathrm{FP}}=2\nu_{\mathrm{F}}=6/(d+2) \qquad 1 \leqslant d<4. \tag{12b}$$

For completeness we note the limit,

$$\lim_{\varepsilon \to 0^+}(\gamma_{\mathrm{RG}}-1)/\phi=\tfrac{1}{4} \tag{12c}$$

so that the logarithmic corrections to scaling in C_n at the critical dimension ($\varepsilon=0$) have the form [43, 44],

$$C_n \sim A_c \mu^n (\ln \mathrm{n})^{1/4} \qquad \mathrm{n} \to \infty. \tag{13}$$

Previous SCF calculations by Kosmas and Freed [36] yield γ (SCF, $d=3$) $=1.2$ which is consistent with (12b) when fluctuation corrections to the SCF potential are included in the theory. They note, however, that higher order calculations could alter this conclusion. Evidently, the 'Flory theory' γ_{FP} is not accurate in low dimensions. Equation (12b) deviates by 50% from the exact result $\gamma(d=1)=1$. The deficiency of (12b) is important in relation to recent studies of SAWs on fractal lattices [17, 18] and percolation clusters [19, 20] (see below).

Given the qualitative inaccuracy of equations (12a) and (12b) it is useful to develop a more accurate estimate of γ for certain applications. Douglas presents an exact geometrical interpretation of γ in a simpler 'excluded volume' problem involving Brownian chains in the proximity of impenetrable boundaries [45]. A heuristic extension of these geometrical arguments, to be presented elsewhere, indicates a simple approximation for the SAW exponent γ,

$$\gamma \approx (d/2)(3\nu-1). \tag{14}$$

Equation (14) is exact in $d=1$, is consistent with RG theory to order ε [see equation (12a)] and is reasonably accurate [21, 39, 40] for $d=3$ where $\nu=0.59$ implies $\gamma=1.16$. However, there is a 7% discrepancy between (14) and the Nienhuis result [12] of $\gamma(d=2)=\frac{43}{32}$. The graph of (14) in figure 2 using ν from (11) has a maximum near $d=2$ as suggested above. The variation of the SAW model exponent γ is contrasted with γ (Ising) which diverges [21] as $d \to 1^+$ and the spherical model γ_s which approaches infinity [32] as $d \to 2^+$ (see equation (6)).

3. Conclusion

We describe the statistics of self-avoiding walks in continuously variable dimension using information gathered from direct enumeration of SAW configurations in integer dimensions ($d=1, 2, \ldots$). The quantities $\langle R_n^2 \rangle$ and C_n are computed in general dimension d and for arbitrary nearest-neighbour interaction. The method is very general and can be applied to related lattice models: spin models, percolation theory, and branched polymers. Baker and Benofy [25] show that the analytic continuation to continuous spatial dimensions employed in our lattice calculations is 'equivalent' to the dimensional continuation used in continuum field theory. This connection permits meaningful comparison between our lattice model estimates for the SAW exponents γ and ν with continuum model RG ε-expansion [3], Borel resummation [21] and Flory SCF computations [33-36, 43].

The Flory SCF value of $\nu_F = 3/(d+2)$, $1 \leqslant d < 4$, is found to agree with our lattice model estimates at low dimensions while the RG ε-expansion theory agrees with our lattice calculations better near four dimensions ($\varepsilon \approx 0^+$). An approximant is introduced which incorporates the precisely known information in low dimensions and the RG predictions for $\varepsilon \approx 0^+$. This expression seems to describe the variation of the lattice SAW exponents over the entire range of spatial dimensionality.

Lattice model calculations of γ in low dimensions ($d \leqslant 2.5$) deviate significantly from analytic predictions based on RG ε-expansion or Flory SCF theory. A consideration of the geometrical significance of γ yields an estimate of $\gamma(d)$ in reasonable accord with lattice model computations. The lattice model values of γ vary non-monotonically in the range $1 < d < 4$ and γ seems to have a maximum near two dimensions. Lattice model estimates of the specific heat and the number of nearest-neighbour contacts of SAWs also exhibit maxima for d in the range of $2 \leqslant d \leqslant 3$, as will be discussed in a subsequent paper focusing on free-energy related SAW properties. The property of SAWs being maximally 'coiled' in intermediate spatial dimensions $2 \leqslant d \leqslant 3$, i.e. having maximal nearest-neighbour contacts, is an important qualitative characteristic of SAWs. Recognition of this property seems to be new.

The maximum in the SAW exponent γ for $d \approx 2$ gives some qualitative insight into recent calculations of γ and ν for SAWs on a family of Sierpinsky gasket lattices [17, 18] having fractal dimensions d_f ranging from 1.5 to 1.75 and into recent estimates of SAW exponents on percolation clusters embedded in $d = 2$ dimensions [19, 20]. *Exact* calculations of ν for Sierpinsky gaskets embedded in $d = 2$ show that an increase in the fractal dimension of the lattice produces a *decrease* of ν and an *increase* of γ [17, 18]. This behaviour is expected from equation (14) and our lattice model computations for SAWs in variable dimension as a consequence of the γ maximum if increasing d_f is taken as qualitatively equivalent to an increase of spatial dimension d. By the same argument we should expect that both ν and γ for SAWs on Menger sponge fractals in $d = 3$ should both decrease with increasing d_f. This prediction could be readily checked. Further, we can obtain some insight into SAWs on percolation clusters embedded in $d = 2$ where ν estimates larger than for the full square lattice are obtained, while γ estimates are nearly unchanged [19, 20]. Such behaviour is expected for dimensions near the maximum of γ ($d \approx 2$). Percolation clusters in $d = 2$ have a fractal dimension near two, $d_f = \frac{91}{48} \approx 2$ [46].

Bhanot *et al* [47] introduce a heuristic relation between spatial dimension d and variable lattice fractal dimension d_f by defining an 'effective dimension' for fractal lattices in terms of an average number of interacting nearest-neighbours. Adopting this definition and using our analytic continuation method should enable estimates to be made for ν and γ of SAWs on fractal lattices. We also expect the variable dimension calculations of γ and ν to be useful in the study of 'dimensional reduction', where the effective spatial dimension varies due to finite size constraints [49].

We plan to extend the present calculations to the $O(m)$ lattice model which should enable non-perturbative calculation of the exponents γ, ν for polymers, the Ising model, Heisenberg model, ..., spherical model. Various 'non-universal' model parameters such as the critical temperature and amplitudes can be conveniently calculated using $1/d$ expansion within the same framework [15, 16]. An investigation of the interplay of spatial dimension and 'spin dimension' m on ν and γ should prove interesting. Calculations are in progress for ν and γ for theta-point and neighbour-avoiding (NAWs) polymers ($\omega \to -\infty$) in variable dimensions [50].

Acknowledgments

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